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Bounded Oscillation of Second-Order Delay Difference Equations of Unstable Type

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Abstract—A series of sufficient conditions are obtained for the bounded oscillation of second-order delay difference equations of unstable type

$$\Delta^2 x_n = \sum_{i=1}^{m} p_i(n) x_{n-k_i}.$$

Some of them are sharp.

Keywords—Bounded oscillation, Difference equation, Unstable type.

1. INTRODUCTION

Consider the second-order linear delay difference equation

$$\Delta^2 x_n = p_n x_{n-k}, \qquad n = 0, 1, 2, \dots, \tag{1.1}$$

and

$$\Delta^2 x_n = \sum_{i=1}^{m} p_i(n) x_{n-k_i}, \qquad n = 0, 1, 2, \dots, \tag{1.2}$$

where $\{p_n\}, \{p_i(n)\}$ are sequences of nonnegative numbers, k, k_i are positives, $i = 1, 2, \dots, m$, and Δ denotes the forward operator $\Delta x_n = x_{n+1} - x_n$.

In [1], it is shown that equation (1.1) always has an unbounded and nonoscillatory solution. Similarly, we may prove that equation (1.2) always has an unbounded and nonoscillatory solution as well. Therefore, we only need to find conditions for all bounded solutions of equation (1.1) or (1.2) to be oscillatory.

Recently, there have been many investigations into the study of delay difference equations. In particular, an extensive literature now exists on the oscillation theory for delay difference equations (we refer to [2–13] and the references cited therein). However, the corresponding

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results on bounded oscillation for (1.1) and (1.2) are relatively scarce in the literature; we only see [1].

In the paper [1], the authors established the following bounded oscillation criterion that if

$$\limsup_{n\to\infty}\sum_{i=0}^{k-1}(i+1)p_{n+i} > 1, \tag{1.3}$$

then every bounded solution of (1.1) oscillates.

In this paper, we will establish a series of bounded oscillation criteria for (1.1) and (1.2) by comparison of (1.1) or (1.2) with a related second-order self-adjoint difference equation and first-order delay difference equations. Of them, some are sharp and improve condition (1.3).

As is customary, a solution $\{x_n\}$ of (1.1) or (1.2) is said to be eventually positive if $x_n > 0$ for all large n, and eventually negative if $x_n < 0$ for all large n. It is said to be oscillatory if it is neither eventually positive nor eventually negative.

2. COMPARISON THEOREMS

To obtain the first theorem in this section, we need the following lemma.

LEMMA 2.1. *Assume that* $\lambda > 0$ *and that*

$$x_n > 0, \qquad \Delta x_n < 0, \qquad \Delta^2 x_n > 0, \qquad n \geq N,$$
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\Delta x_n = 0,$$

and

$$\Delta^2 x_n \geq \lambda^2 x_n, \qquad \text{for } n \geq N.$$

Then

$$\Delta x_n + \lambda x_n \leq 0, \qquad \text{for } n \geq N.$$

PROOF. Set $z_n = \Delta x_n + \lambda x_n$. Then

$$\Delta z_n - \lambda z_n = \Delta^2 x_n - \lambda^2 x_n \geq 0, \qquad n \geq N.$$

This implies that $\{z_n(1+\lambda)^{-n}\}$ is nondecreasing. Since $x_n \to 0$, $\Delta x_n \to 0$ as $n \to \infty$, we have $z_n \to 0$ as $n \to \infty$. Thus, $z_n \leq 0$ for $n \geq N$, i.e.,

$$\Delta x_n + \lambda x_n \leq 0, \qquad \text{for } n \geq N.$$

The proof is complete.

The next two theorems are the main results in this section.

THEOREM 2.1. *Assume that* $0 < \epsilon < 1$ *and that for large* n,

$$a_n \equiv p_n - \frac{4k^k}{(k+2)^{k+2}} \geq 0. \tag{2.1}$$

Then every solution of the second-order self-adjoint difference equation

$$\Delta^2 y_n + (1-\epsilon)\left(\frac{k+2}{k}\right)^{k+1} a_n y_n = 0, \qquad n = 0,1,2,\dots, \tag{2.2}$$

oscillating implies that every bounded solution of (1.1) oscillates.

PROOF. For the sake of contradiction, assume that (1.1) has an eventually positive bounded solution $\{x_n\}$. It is not difficult to show that there exists a positive integer N_0 such that

$$a_n \geq 0, \quad x_n > 0, \quad \Delta x_n < 0, \quad \text{and} \quad \Delta^2 x_n > 0, \qquad n \geq N_0 - k, \tag{2.3}$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \Delta x_n = 0. \tag{2.4}$$

Set $v_n = x_n[(k+2)/k]^n$. Then $v_n > 0$ for $n \geq N_0 - k$, and from (1.1) and (2.1), we have

$$\Delta^2 \left[v_n - \frac{4}{k^2} \sum_{i=1}^{k} (k+1-i) v_{n-i} \right] = \left(\frac{k+2}{k} \right)^{k+2} a_n v_{n-k}, \qquad n \geq N_0. \tag{2.5}$$

Let

$$u_n = v_n - \frac{4}{k^2} \sum_{i=1}^{k} (k+1-i) v_{n-i}, \qquad n \geq N_0. \tag{2.6}$$

Then,

$$\Delta^2 u_n = \left(\frac{k+2}{k} \right)^{k+2} a_n v_{n-k}, \qquad n \geq N_0. \tag{2.7}$$

By (2.7), the decreasing nature of $\{x_n\}$, and the identity

$$1 + 2x + 3x^2 + \cdots + kx^{k-1} \equiv \frac{1-(k+1)x^k + kx^{k+1}}{(1-x)^2}, \qquad \text{for } x \neq 1,$$

we have for $n \geq N_0$,

$$\begin{aligned}
u_n &= v_n - \frac{4}{k^2} \sum_{i=1}^{k} (k+1-i) v_{n-i} \\
&= x_n \left(\frac{k+2}{k} \right)^n - \frac{4}{k^2} \sum_{i=1}^{k} (k+1-i) x_{n-i} \left(\frac{k+2}{k} \right)^{n-i} \\
&\leq x_n \left[\left(\frac{k+2}{k} \right)^n - \frac{4}{k^2} \sum_{i=1}^{k} (k+1-i) \left(\frac{k+2}{k} \right)^{n-i} \right] \\
&= -\left(\frac{k}{k+2} \right)^k v_n < 0,
\end{aligned}$$

which together with (2.7) implies that

$$u_n < 0, \quad \Delta u_n < 0, \quad \text{and} \quad \Delta^2 u_n \geq 0, \qquad n \geq N_0. \tag{2.8}$$

Next, we will prove that for large n,

$$v_{n-k} \geq -\frac{(1-\epsilon)k}{k+2} u_n. \tag{2.9}$$

From (1.1), (2.1), and (2.3), we have

$$\Delta^2 x_n \geq \frac{4k^k}{(k+2)^{k+2}} x_n, \qquad n \geq N_0. \tag{2.10}$$

By Lemma 2.1, it follows from (2.10), (2.3), and (2.4) that

$$\Delta x_n + \frac{2k^{k/2}}{(k+2)^{(k+2)/2}} x_n \leq 0, \qquad n \geq N_0.$$

Consequently, we have

$$x_{n-k} \geq (1-\lambda_0)^{-k} x_n, \qquad n \geq N_0 + k, \tag{2.11}$$

where

$$\lambda_0 = \frac{2k^{k/2}}{(k+2)^{(k+2)/2}}.$$

Substituting (2.11) into (1.1) and using (2.1), we obtain

$$\Delta^2 x_n \geq \lambda_0^2 (1-\lambda_0)^{-k} x_n, \qquad n \geq N_0.$$

Similar to (2.11), we can conclude from the above, (2.3), and (2.4) that

$$x_{n-k} \geq (1-\lambda_1)^{-k} x_n, \qquad n \geq N_0 + 2k,$$

where

$$\lambda_1 = \lambda_0 (1-\lambda_0)^{-k/2}.$$

By induction, one can easily show that

$$x_{n-k} \geq (1-\lambda_j)^{-k} x_n, \qquad n \geq N_0 + (j+1)k, \tag{2.12}$$

where

$$\lambda_j = \lambda_0 (1-\lambda_{j-1})^{-k/2}, \qquad j = 1, 2, \dots. \tag{2.13}$$

It is not difficult to verify that

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \frac{2}{k+2} \tag{2.14}$$

and

$$\lim_{j \to \infty} \lambda_j = \frac{2}{k+2}. \tag{2.15}$$

Set

$$R(\theta) = \frac{1 + (1-2\theta)[1 + 2(1-\theta)/k]^k}{(1-\theta)^2}, \qquad 0 < \theta < 1. \tag{2.16}$$

Then $\lim_{\theta \to 1^-} R(\theta) = 2(1+k)/k$. Let $\theta_0 \in (0,1)$ such that

$$2 - \epsilon < R(\theta_0) < \frac{(2-\epsilon)k + 2}{(1-\epsilon)k}. \tag{2.17}$$

In view of (2.14) and (2.15), there exists a positive integer N such that

$$\frac{2\theta_0}{k+2} < \lambda_j < \frac{2}{k+2}, \qquad j \geq N. \tag{2.18}$$

Hence, it follows from (2.12) and (2.13) that

$$x_{n-k} \geq \left[\frac{2\theta_0}{(k+2)\lambda_0}\right]^2 x_n, \qquad n \geq N_0 + Nk. \tag{2.19}$$

In view of Lemma 2.1, from (2.3), (2.4), and (2.19), we have

$$\Delta x_n + \frac{2\theta_0}{k+2} x_n \leq 0, \qquad n \geq N_0 + Nk.$$

Consequently, we obtain

$$x_{n-i} \leq x_{n-k} \left[\frac{k + 2(1-\theta_0)}{k+2}\right]^{k-i}, \qquad 1 \leq i \leq k, \quad n \geq N_0 + (N+1)k. \tag{2.20}$$

Substituting (2.20) into (2.6), we have for $n \geq N_1 = N_0 + (N+1)k$,

$$\begin{aligned} u_n &= v_n - \frac{4}{k^2} \sum_{i=1}^{k} (k+1-i) v_{n-i} \\ &= v_n - \frac{4}{k^2} \sum_{i=1}^{k} (k+1-i) \left(\frac{k+2}{k}\right)^{n-i} x_{n-i} \\ &\geq v_n - \frac{4}{k^2} \left(\frac{k+2}{k}\right)^{n-k} x_{n-k} \sum_{i=1}^{k} (k+1-i) \left[\frac{k+2(1-\theta_0)}{k+2}\right]^{k-i} \\ &= v_n - R(\theta_0) v_{n-k}. \end{aligned}$$

From this and (2.17), we obtain

$$\frac{(2-\epsilon)k+2}{(1-\epsilon)k} v_{n-k} \geq v_n - u_n, \qquad n \geq N_1. \tag{2.21}$$

It follows that

$$v_{n-k} \geq -\sum_{j=0}^{\infty} \frac{[(1-\epsilon)k]^{j+1} u_{n+jk}}{[(2-\epsilon)k+2]^{j+1}} \geq -\frac{(1-\epsilon)k}{k+2} u_n, \qquad n \geq N_1. \tag{2.22}$$

This shows that (2.9) holds. From (2.7) and (2.22), we have

$$\Delta^2 u_n \geq -(1-\epsilon) \left(\frac{k+2}{k}\right)^{k+1} a_n u_n, \qquad n \geq N_1. \tag{2.23}$$

Set $y_n = -u_n$. Then $y_n > 0$ for $n \geq N_1$ and

$$\Delta^2 y_n + (1-\epsilon) \left(\frac{k+2}{k}\right)^{k+1} a_n y_n \leq 0, \qquad n \geq N_1. \tag{2.24}$$

This shows that inequality (2.24) has an eventually positive solution. By a known theorem in [3,4], the corresponding equation (2.2) also has eventually positive solution, leading to a contradiction. The proof is complete.

THEOREM 2.2. *Assume that $p_n < 1$ for large n. Then every solution of the difference equation*

$$\Delta y_n + \frac{1}{1+p_{n+k+1}} \sum_{i=0}^{k-1} \left(\prod_{j=n-k+i+1}^{n+1} \frac{1+p_{j+k}}{1-p_{j+k-1}} \right) p_{n+i} y_{n-k+i} = 0, \qquad n = 0, 1, 2, \dots \tag{2.25}$$

oscillating implies that every bounded solution of (1.1) oscillates.

PROOF. For the sake of contradiction, assume that (1.1) has a bounded eventually positive solution $\{x_n\}$. It is not difficult to show that there exists a positive integer N such that

$$p_n < 1, \quad x_n > 0, \quad \Delta x_n < 0, \quad \text{and} \quad \Delta^2 x_n > 0, \qquad n \geq N, \tag{2.26}$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \Delta x_n = 0. \tag{2.27}$$

Hence, summing (1.1) from n to $n+k+1$, we have

$$\Delta x_n + \sum_{i=0}^{k+1} p_{n+i} x_{n-k+i} \leq 0, \qquad n \geq N. \tag{2.28}$$

Set

$$y_n = x_n \prod_{i=N}^{n} \frac{1+p_{i+k}}{1-p_{i+k-1}}, \qquad \text{for } n \geq N.$$

Then, from (2.28), we have

$$\Delta y_n + \frac{1}{1+p_{n+k+1}} \sum_{i=0}^{k-1} \left(\prod_{j=n-k+i+1}^{n+1} \frac{1+p_{j+k}}{1-p_{j+k-1}} \right) p_{n+i} y_{n-k+i} \leq 0, \qquad n \geq N. \tag{2.29}$$

This shows that inequality (2.29) has an eventually positive solution. In view of Theorem 1 in [12], the corresponding equation (2.25) also has an eventually positive solution. This is a contradiction, and so the proof is complete.

3. OSCILLATION CRITERIA FOR EQUATION (1.1)

In this section, we give some sufficient conditions for bounded oscillation and nonoscillation of equation (1.1) by employing Theorems 2.1 and 2.2, and existing oscillation criteria for equations (2.2) and (2.25).

THEOREM 3.1.

(i) *If*

$$\liminf_{n\to\infty} \left[\left(p_n - \frac{4k^k}{(k+2)^{k+2}} \right) n^2 \right] > \frac{1}{4} \left(\frac{k}{k+2} \right)^{k+1}, \tag{3.1}$$

then every bounded solution of (1.1) oscillates.

(ii) *If the vector $(p_{n+1}, \dots, p_{n+k}) \neq 0$ for large n and eventually*

$$\left(p_n - \frac{4k^k}{(k+2)^{k+2}} \right) n^2 \leq \frac{1}{4} \left(\frac{k}{k+2} \right)^{k+1}, \tag{3.2}$$

then (1.1) has a bounded eventually positive solution.

PROOF.

(i) It follows from (3.1) that there exists $\epsilon \in (0,1)$ such that

$$\liminf_{n\to\infty} \left[\left(p_n - \frac{4k^k}{(k+2)^{k+2}} \right) n^2 \right] > \frac{1}{4(1-\epsilon)} \left(\frac{k}{k+2} \right)^{k+1}. \tag{3.3}$$

In view of Lemma 3 in [11], (3.3) implies that every solution of (2.2) oscillates. Hence, by Theorem 2.1, every bounded solution of (1.1) also oscillates.

(ii) Let

$$q_n = \frac{4k^k}{(k+2)^{k+2}} + \frac{k^{k+2}}{(k+2)^{k+2}\sqrt{n-k}} \times \left[\frac{\sqrt{n}-\sqrt{n+2}}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n+2}\right)} + \frac{4\left(\sqrt{n+1}-\sqrt{n-k}\right)}{k\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n}+\sqrt{n-k}\right)} \right],$$

for $n > k$. We verify by direct calculation that for $n > k$,

$$\begin{aligned} q_n - \frac{4k^k}{(k+2)^{k+2}} &= \frac{k^{k+2}}{(k+2)^{k+2}\sqrt{n-k}} \\ &\quad \times \left[\frac{\sqrt{n}-\sqrt{n+2}}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n+2}\right)} + \frac{4\left(\sqrt{n+1}-\sqrt{n-k}\right)}{k\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n}+\sqrt{n-k}\right)} \right] \\ &\geq \frac{2k^{k+1}}{(k+2)^{k+1}\sqrt{n-k}\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n}+\sqrt{n-k}\right)\left(\sqrt{n+1}+\sqrt{n-k}\right)} \\ &\geq \frac{1}{4n^2} \left(\frac{k}{k+2} \right)^{k+1}. \end{aligned}$$

It follows from (3.2) that there exists an integer $N > k$ such that

$$p_n \le \frac{4k^k}{(k+2)^{k+2}} + \frac{1}{4n^2}\left(\frac{k}{k+2}\right)^{k+1} \le q_n, \qquad n \ge N. \tag{3.4}$$

On the other hand, it is easy to see that the function $z_n = \sqrt{n}(k/k+2)^n$ is a positive bounded solution of the following equation:

$$\Delta^2 z_n = q_n z_{n-k}, \qquad n > k. \tag{3.5}$$

In view of (3.4) and (3.5), it is easy to show that (1.1) has a bounded eventually positive solution. The proof is complete.

From Theorem 3.1, we have the following corollary.

COROLLARY 3.1.

(i) *If*

$$\liminf_{n\to\infty} p_n > \frac{4k^k}{(k+2)^{k+2}}, \tag{3.6}$$

then every bounded solution of (1.1) oscillates.

(ii) *If the vector* $(p_{n+1}, \dots, p_{n+k}) \ne 0$ *for large* n *and eventually*

$$p_n \le \frac{4k^k}{(k+2)^{k+2}}, \tag{3.7}$$

then (1.1) has a bounded eventually positive solution.

THEOREM 3.2. *Assume that* $p_n < 1$ *for large* n, *and that*

$$\liminf_{n\to\infty}\left[\sum_{i=0}^{k-1}\left(\frac{k-i+1}{k-i}\right)^{k-i+1}\sum_{j=n+1}^{n+k-i}\frac{p_{j+i}}{1+p_{j+k+1}}\prod_{s=j-k+i+1}^{j+1}\frac{1+p_{s+k}}{1-p_{s+k-1}}\right] > 1. \tag{3.8}$$

Then every bounded solution of (1.1) oscillates.

Theorem 3.2 is a immediate corollary of Theorem 2.2 and Corollary 4 in [7].

THEOREM 3.3. *Assume that* $p_n < 1$ *for large* n, *and that*

$$\limsup_{n\to\infty}\left[\sum_{i=0}^{k-1}\sum_{j=n}^{n+k-i}\frac{p_{j+i}}{1+p_{j+k+1}}\prod_{s=j-k+i+1}^{j+1}\frac{1+p_{s+k}}{1-p_{s+k-1}}\right] > 1. \tag{3.9}$$

Then every bounded solution of (1.1) oscillates.

PROOF. By Theorem 2.2, it suffices to prove that every solution of (2.25) oscillates. Suppose the contrary. Then (2.25) may have an eventually positive and nonincreasing solution $\{y_n\}$. Let N be sufficiently large that

$$y_{n-k} > 0, \quad y_{n+1} - y_n \le 0, \qquad \text{for } n \ge N.$$

Summing (2.25) from N to ∞, we obtain

$$\begin{aligned}
y_N &\ge \sum_{n=N}^{\infty}\frac{1}{1+p_{n+k+1}}\sum_{i=0}^{k-1}\left(\prod_{j=n-k+i+1}^{n+1}\frac{1+p_{j+k}}{1-p_{j+k-1}}\right)p_{n+i}y_{n-k+i}\\
&\ge \sum_{i=0}^{k-1}\sum_{n=N}^{N+k-i}\frac{1}{1+p_{n+k+1}}\left(\prod_{j=n-k+i+1}^{n+1}\frac{1+p_{j+k}}{1-p_{j+k-1}}\right)p_{n+i}y_{n-k+i}\\
&\ge y_N\sum_{i=0}^{k-1}\sum_{n=N}^{N+k-i}\frac{p_{n+i}}{1+p_{n+k+1}}\left(\prod_{j=n-k+i+1}^{n+1}\frac{1+p_{j+k}}{1-p_{j+k-1}}\right).
\end{aligned}$$

It follows that

$$\sum_{i=0}^{k-1} \sum_{n=N}^{N+k-i} \frac{p_{n+i}}{1+p_{n+k+1}} \left(\prod_{j=n-k+i+1}^{n+1} \frac{1+p_{j+k}}{1-p_{j+k-1}} \right) \le 1.$$

Taking the limit superior as $N \to \infty$ in the above inequality, we obtain

$$\limsup_{N\to\infty} \left[\sum_{i=0}^{k-1} \sum_{n=N}^{N+k-i} \frac{p_{n+i}}{1+p_{n+k+1}} \prod_{j=n-k+i+1}^{n+1} \frac{1+p_{j+k}}{1-p_{j+k-1}} \right] \le 1,$$

which contradicts (3.9). The proof is complete.

THEOREM 3.4. *Assume that*

$$\limsup_{n\to\infty} \sum_{i=0}^{k} (i+1) p_{n+i} > 1. \tag{3.10}$$

Then every bounded solution of (1.1) oscillates.

PROOF. For the sake of contradiction, assume that (1.1) has a bounded eventually positive solution $\{x_n\}$. It is not difficult to show that there exists a positive integer N such that

$$x_n > 0, \quad \Delta x_n < 0, \quad \text{and} \quad \Delta^2 x_n > 0, \qquad n \ge N.$$

Summing (1.1) from $n \ge N$ to $n+k$, we obtain

$$\Delta x_n + \sum_{i=0}^{k} p_{n+i} x_{n-k+i} \le 0, \qquad n \ge N.$$

Summing (1.11) from $n \ge N$ to $n+k$ again, we obtain

$$\begin{aligned} x_n &\ge \sum_{i=0}^{k} \sum_{j=n}^{n+k} p_{j+i} x_{j-k+i} \ge \sum_{i=0}^{k} \sum_{j=n}^{n+k-i} p_{j+i} x_{j-k+i} \\ &\ge x_n \sum_{i=0}^{k} \sum_{j=n}^{n+k-i} p_{j+i} = x_n \sum_{i=0}^{k} (i+1) p_{n+i}. \end{aligned}$$

It follows that

$$\sum_{i=0}^{k} (i+1) p_{n+i} \le 1, \qquad n \ge N.$$

Taking the limit superior as $n \to \infty$ in the above inequality, we obtain

$$\limsup_{n\to\infty} \sum_{i=0}^{k} (i+1) p_{n+i} \le 1,$$

which contradicts (3.10) and so the proof is complete.

REMARK 3.1. Clearly, both (3.9) and (3.10) improve condition (1.3).

4. OSCILLATION CRITERIA FOR EQUATION (1.2)

THEOREM 4.1. *Assume that*

$$\liminf_{n\to\infty} \sum_{i=1}^{m} \frac{(k_i+2)^{k_i+2}}{4k_i^{k_i}} p_i(n) > 1. \tag{4.1}$$

Then every bounded solution of (1.2) oscillates.

PROOF. For the sake of contradiction, assume that (1.2) has an eventually positive bounded solution $\{x_n\}$. It is not difficult to show that there exists a positive integer N such that

$$x_n > 0, \quad \Delta x_n < 0, \quad \text{and} \quad \Delta^2 x_n > 0, \qquad n \geq N, \tag{4.2}$$

$$\frac{4}{(k+2)^2e^2} < \sum_{i=1}^{m} p_i(n) \leq m, \qquad n \geq N, \tag{4.3}$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \Delta x_n = 0. \tag{4.4}$$

Then from (1.2), (4.2), and (4.3), we have

$$\Delta^2 x_n \geq \frac{4}{(k+2)^2e^2} x_{n-1}, \qquad n \geq N, \tag{4.5}$$

and

$$\Delta^2 x_n \leq m x_{n-k}, \qquad n \geq N, \tag{4.6}$$

where $k = \max\{k_1, k_2, \ldots, k_m\}$. Define a set Λ as follows:

$$\Lambda = \left\{\lambda > 0 : \Delta^2 x_n \geq \lambda^2 x_n \text{ eventually}\right\}. \tag{4.7}$$

It is easy to see that $2/(k+2)e \in \Lambda$, i.e., Λ is nonempty. We shall show that Λ is bounded above. In fact, it follows from (4.2) and (4.5) that

$$\Delta x_n + \frac{4}{(k+2)^2e^2} x_{n-1} \leq 0, \qquad n \geq N, \tag{4.8}$$

and so,

$$x_{n-k} \leq \frac{(k+2)^2e^2}{4} x_{n-k+1} \leq \cdots \leq \frac{(k+2)^{2k}e^{2k}}{4^k} x_n, \qquad n \geq N + k.$$

Substituting this into (4.6), we obtain

$$\Delta^2 x_n \leq \frac{(k+2)^{2k}e^{2k}}{4^k} m x_n, \qquad n \geq N,$$

which implies that $(k+2)^k e^k \sqrt{m}/2k \notin \Lambda$, i.e., Λ is bounded above.

Set $\lambda_0 = \sup \Lambda$. Then $\lambda_0 \in (0, (k+2)^k e^k \sqrt{m}/2^k]$. For any $\alpha \in (0,1)$, there exists a positive integer $N_1 = N_1(\alpha) \geq N$ such that

$$\Delta^2 x_n \geq (\alpha\lambda_0)^2 x_n, \qquad n \geq N_1 - k. \tag{4.9}$$

In view of Lemma 2.1, we have

$$\Delta x_n + \alpha\lambda_0 x_n \leq 0, \qquad n \geq N_1 - k. \tag{4.10}$$

It follows that

$$x_{n-k_i} \geq (1-\alpha\lambda_0)^{-k_i} x_n, \qquad i = 1, 2, \ldots, m, \quad n \geq N_1.$$

Substituting this into (1.2), we obtain

$$\Delta^2 x_n \geq x_n \sum_{i=1}^{m} p_i(n)(1-\alpha\lambda_0)^{-k_i}, \qquad n \geq N_1,$$

which implies that

$$\inf_{n\geq N_1} \left\{\sum_{i=1}^{m} p_i(n)(1-\alpha\lambda_0)^{-k_i}\right\} \leq \lambda_0^2,$$

and so,

$$\inf_{n\geq N_1} \left\{\sum_{i=1}^{m} \frac{(k_i+2)^{k_i+2}}{4k_i^{k_i}} p_i(n)\right\} \leq \alpha^{-2}.$$

Letting $\alpha \to 1$ and $N_1 \to \infty$, we have

$$\liminf_{n\to\infty} \sum_{i=1}^{m} \frac{(k_i+2)^{k_i+2}}{4k_i^{k_i}} p_i(n) \leq 1,$$

which contradicts (4.1). The proof is complete.

THEOREM 4.2. *Assume that*

$$\liminf_{n\to\infty}\left[\sum_{i=1}^{m}\sum_{j=0}^{k_i-1}\left(\frac{k_i-j+1}{k_i-j}\right)^{k_i-j+1}\sum_{s=n+1}^{n+k_i-j}p_i(s+j)\right]>1. \tag{4.11}$$

Then every bounded solution of (1.2) oscillates.

PROOF. If not, there is a bounded nonoscillatory solution $\{x_n\}$ of (1.2), which, without loss of generality, we can assume to be eventually positive. It is easy to show that there exists a positive integer N such that (4.2) holds. Summing (1.2) from $n \geq N$ to ∞, we obtain

$$\Delta x_n+\sum_{i=1}^{m}\sum_{j=0}^{k_i}p_{n+j}x_{n-k_i+j}\leq 0, \qquad n\geq N. \tag{4.12}$$

This shows that (4.12) has an eventually positive solution. But, in view of Corollary 4 in [7], (4.11) implies that (4.12) has no eventually positive solutions. This contradiction complete the proof.

Similar to the proof of Theorem 3.4, we can prove the following.

THEOREM 4.3. *Assume that*

$$\limsup_{n\to\infty}\left[\sum_{i=1}^{m}\sum_{j=0}^{k_i}(j+1)p_i(n+j)\right]>1. \tag{4.13}$$

Then every bounded solution of (1.2) oscillates.

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